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A SIMPLE ALGORITHM FOR ROTATING THE PLANE OF AN ORBIT IN THE CENTRAL AND NONCENTRAL GRAVITATIONAL FIELD OF THE EARTH

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A SIMPLE ALGORITHM FOR ROTATING THE PLANE OF AN ORBIT IN THE CENTRAL AND NONCENTRAL GRAVITATIONAL FIELD OF THE EARTH

V.V. Veselovskiy

ABSTRACT: The problem of the repeated passage of a satellite over a given point on the terrestrial surface is discussed. A simple algorithm is proposed for rotating the plane of the osculating orbit which is initially described as an example in the central gravitational field of the earth and then extended to the noncentral field, which is taken into account with an accuracy up to the first harmonic.

The functional is derived on which is based the switching on and switching off of propulsion and the accuracy of the thrust cut-off is checked for the size of the miss during repeated passage of the satellite. The thrust is assumed to be limited in magnitude and acts for a specific finite interval of time, and the satellite is assumed to be in the form of a material point of variable mass.

It is shown that for certain assumptions, the satellite's equations of motion are separable. An estimate of the error in the solution for the simplified problem is carried out.

Statement of the Problem

1. The problem of the repeated passage of a satellite over a given point /3* on the terrestrial surface is discussed. A rotation is carried out by a force normal to the plane of the osculating orbit, which is assumed to be near-circular. The times of switching on and switching off the thrust are determined from considerations of an accurate passage over a given point. By "accuracy of passage" is understood the minimum angle between the radius vectors of the point and the satellite. A functional is initially derived for a central field and then for a noncentral field; the gravitational field of the earth is taken into account with an accuracy up to the first harmonic.

The calculations show that neglect of the variability of the satellite's mass causes an error in determining the position of the orbit's plane as large as several degrees, which is objectionable.

* Numbers in the margin indicate pagination in the foreign text.

An algorithm for solving the problem reduces to the following. The satellite's spatial equations of motion are solved. In the computational process all the variables of the active trajectory at any time are known. Taking them as the initial conditions of the satellite's free-flight trajectory, it is possible to derive its parameters from which the corresponding functional can be formulated. The latter depends, in the final analysis, on the parameters of the active trajectory and on the parameters of the free-flight trajectory. Its rezeroing takes place only under the conditions of the existence of a solution /4
of the encounter problem. At this time, the transition of active flight to free flight takes place, i.e., thrust cut-off occurs. The time of rezeroing the functional should be determined extremely accurately. The trajectories of a satellite's free motion are known for a central field, but for the case of a noncentral field, relatively simple analytic expressions are derived which are brought into the corresponding functional. An estimate is given in this paper of the error of these expressions which corresponds to the assumed model of the gravitational field. Such an approach to the solution of the problem permitted simplifying as much as possible the algorithm for rotating the orbit's plane and simplifying to a minimum the expenditures of machine time for carrying out all the numerical operations.

The motion of the object around the center of mass was not taken into account in the present work, and the errors introduced by the initial data and the instrumental errors (both determined and accidental) of the on-board and ground-based control system were not considered.

In Appendix 3, the systems of equations are presented according to which the computation of the algorithm for rotating the orbit's plane in a noncentral field is carried out. For comparison, the control algorithm in a central field is given in Appendix 1.

The justification for replacing the complete system of equations of motion of the satellite by the simplified set with the corresponding limitations which occur in a given problem is given in Appendix 2.

The fact that the derived algorithm can be reproduced on an on-board computer which operates in real time is of significant value. /5

Algorithm for Rotating the Orbital Plane in a Central Field

2. The equations of spatial motion of a satellite in osculating variables are described in a well-known manner. One can, for example, transform them to the following equivalent form [1, 2].

$$
\left.\begin{array}{l}
1.\ y'' + y - \dfrac{1}{p} = -\dfrac{1}{py}\left[K_r + \dfrac{y'}{y} K_v\right], \\
2.\ p' = \dfrac{2}{y} K_v, \\
3.\ i' = \dfrac{K_z}{py} \cos u, \\
4.\ W' = -\dfrac{K_z}{py} \operatorname{ctg} i \cdot \sin u, \\
5.\ \Omega' = \dfrac{K_z}{py} \operatorname{cosec} i \cdot \sin u, \\
6.\ u = v + w
\end{array}\right\} \qquad (2.1)
$$

where it is assumed that

1) = d/dv is the vector of the perturbing accelerations,

v is the true anomaly,

k_r, k_v, k_z are the projections of the transfer vector $\bar{k}$ onto the orbital axes directed along the radius vector, along the perpendicular to it in the osculating plane in the direction of motion, and along the perpendicular to the plane,

$$\bar{K} = \frac{\bar{j}}{\mu y^2}, \qquad (2.2)$$

where

$\bar{j}$ is the vector of the perturbing accelerations, /6

n is the earth's gravitational constant,

y = 1/r is the inverse of the radius vector,

p is the orbit's parameter,

W is the angle determining the position of the line of apsides, and

Ω is the longitude of the ascending node.

The variable v, as is well known, is associated with the time t by the following equation

$$t' = \frac{1}{\sqrt{\mu p}\, y^2} \qquad (2.3)$$

In this case, when the satellite is taken to be a material point of variable mass, the transfer vector can be written in the form

$$\bar{K} = \frac{\bar{K}_0}{1 - B\tau}, \qquad (2.4)$$

where

$\bar{k}_o$ is the initial (at $\tau = 0$) transfer;

B is the coefficient which takes into account the variation in mass,

$$\tau = \begin{cases} 0 & t < \tau_{on} \\ t - t_{on} & \tau_{on} \leq t \leq \tau_{off} \\ t_{off} - t_{on} & t > \tau_{off} \end{cases} \qquad (2.5)$$

t_{on} and t_{off} are the times of switching on and switching off the thrust, respectively.

We note that equations 1 and 2 in (2.1) represent the motion of a satellite in the plane of the osculating orbit and equations 3 to 5 represent the position of this plane in space. In the case of motion in a central field of attraction the system (2.1) is separable in the case where $k_r = k_v = 0$; the values of p and y are known for elliptic motion /7

$$py = 1 + e\cos(v - v_p), \qquad (2.6)$$

where

e is the eccentricity of the elliptic orbit and

v_p corresponds to the orbit's time of perigess passage.

3. Let the satellite (S) move in a central field, and at some specified time its position over a specific point on the terrestrial surface (P) is known,

and it is required that during its subsequent return, it again pass over this point on the terrestrial surface. For this we apply the transfer k_z, whose times of switching on and switching off are determined by the relation (2.5); the satellite is considered to be a material point of variable mass (during operation of the propulsion). Such a problem does not have an analytic solution; therefore, we attempt to formulate an algorithm for finding the quantities τ_{on} and τ_{off}; for simplicity we consider the orbit to be circular.

We consider Figure 1, on which it is illustrated that:

S, P is the position of the satellite and the selected point on the terrestrial surface at the current instant of time;

B is the point of intersection of S and P;

φ_o is the latitude of the point P, a constant quantity;

λ_P, λ_S, λ_B are the longitude of the points P, S, and B, respectively;

i_1 is the angle of inclination of the plane of the initial orbit;

i is the current angle of inclination of the orbital plane, and

Ω_1, Ω are the angles of right ascension for the initial and current orbital planes, respectively. /8

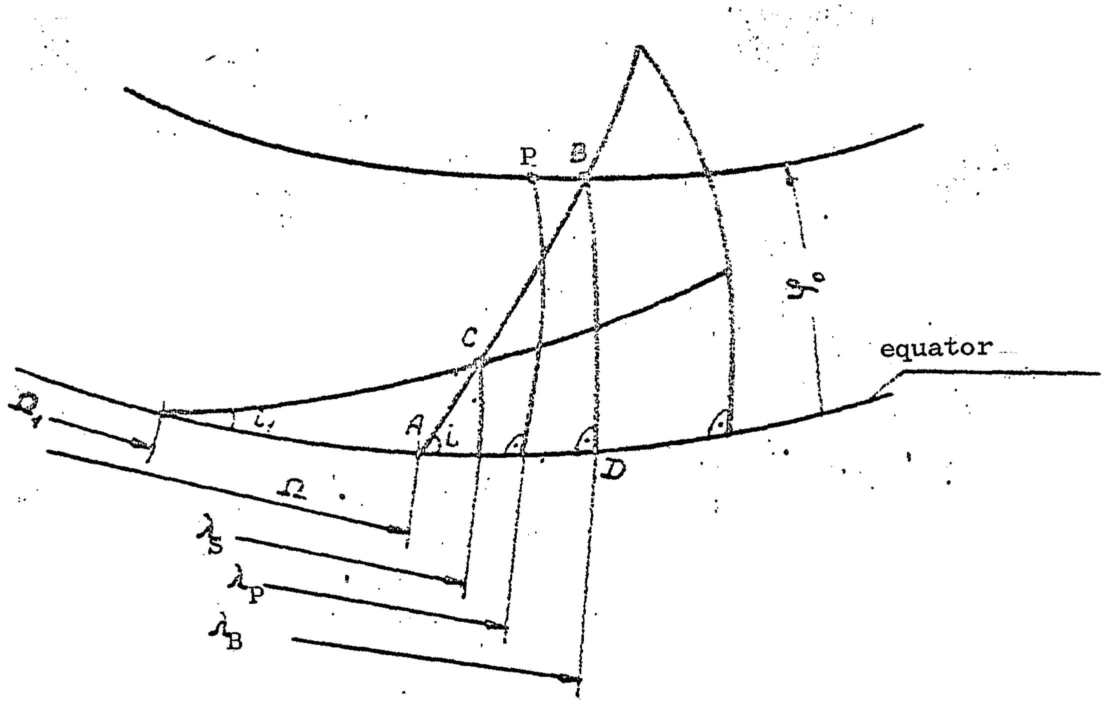

Figure 1. Geometry of the motion.

From consideration of the spherical right-angle triangle ABD, the following relations hold:

$$\sin(\lambda_B - \Omega) = \frac{tg\,\varphi_0}{tg\,i}, \tag{2.7}$$

$$\sin(u + \Delta u_y) = \frac{\sin\varphi_0}{\sin i}, \tag{2.8}$$

where

λ_B is the longitude of the point B, assumed to be the instant of encounter, and Δu_y corresponds to the arc CB along the orbit.

Knowledge of the quantity Δu_y in the case of the problem of a central field uniquely determines the value of the argument Δv_y for the free-flight trajectory (which is proportional to the time of prediction Δt_y), thus determining the instant of the encounter, since

$$u = v + w,$$

and w = const after the cut-off for motion in a central field.

4. The equations which determine the position of the plane of the osculating orbit in (2.1) possess the inadequacy that at $i = 0$ there appears a singularity in the solution. In addition, it is more convenient when calculating on a computer to get rid of as much of the trigonometric functions as possible. We use [2] and perform a replacement of variables:

$$\left.\begin{aligned} \gamma_1 &= \sin i \cdot \sin u, \\ \gamma_2 &= \sin i \cdot \cos u, \\ \gamma_3 &= \cos i, \end{aligned}\right\} \tag{2.9}$$ ∠9

as a result of this the system of equations assumes a form suitable not only for analysis but also for computations. We note immediately that

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{2.10}$$

We obtain in advance a series of useful equations. It follows from spherical trigonometry and the Eqs. (2.7) and (2.8) that

$$\left.\begin{aligned} \sin\varphi &= \sin u \sin i, \\ \operatorname{tg}(\lambda-\Omega) &= \operatorname{tg} u \cos i, \end{aligned}\right\} \tag{2.11}$$

where

φ, λ are the latitude and longitude, respectively,

Ω, u, i are the angles which determine the position of the satellite in the system of coordinates associated with the orbital plane. Combining (2.9) and (2.11), we obtain

$$\gamma_1 = \sin i \sin u = \sin i \frac{\sin\varphi}{\sin i} = \sin\varphi, \tag{2.12}$$

$$\gamma_3 = \cos i = \frac{\operatorname{tg}(\lambda-\Omega)}{\operatorname{tg} u}, \tag{2.13}$$

$$\frac{\gamma_1}{\gamma_2} = \operatorname{tg} u. \tag{2.14}$$

From (2.12) and (2.13)

$$\operatorname{tg}(\lambda-\Omega) = \frac{\gamma_1}{\gamma_2}\gamma_3. \tag{2.15}$$

By analogy with (2.15) we have /10

$$\operatorname{tg}(\lambda_B - \Omega) = \frac{\gamma_{1z}\gamma_3}{\gamma_2}, \tag{2.16}$$

where

$\gamma_{1z} = \sin\varphi_o$ is the specified latitude of the point P.

5. From consideration of Figure 1, it is possible to derive equations for the functional of the thrust cut-off in a central field. At the switching on of the thrust, the position of the satellite (S) is determined at each current time as a result of solving the system of equations. With the help of the relations (2.7) and (2.8), taking into account the fact that after the cut-off w = const., we derive the size of the prediction of Δv_y[1] and the longitude λ_B of

1. Δv_y is the value of Δv when the functional is rezeroed.

the point of encounter B. The position of the point P at each time is also known (this would be the current position of the point B); it is determined from the following expression:

$$\lambda_P = \lambda_0 + \frac{\Omega_3}{\dot{\nu}_0} v, \tag{2.17}$$

where

ν_o is the angular velocity of the satellite and

$\lambda_P = \lambda_o$ at $v = 0$.

It is completely obvious that in order to have an encounter, it is necessary that the points P and S pass through the point B at the very same time, and this indicates that the following functional should be equal to zero at the current time

$$J = \lambda_B - \lambda_P - \frac{\Omega_3}{\dot{\nu}_0} \Delta v_y = 0. \tag{2.18}$$

The central angle Δ between the radius vectors of the points $S(\bar{r}_s^{\,0})$ and $P(\bar{r}_y^{\,0})$ is determined in the following manner

$$\bar{r}_s^{\,0} = \{\cos\varphi_s \cos\lambda_s, \cos\varphi_s \sin\lambda_s, \sin\varphi_s\}, \tag{2.19}$$ /11

$$\bar{r}_y^{\,0} = \{\cos\varphi_0 \cos\lambda_P, \cos\varphi_0 \sin\lambda_P, \sin\varphi_0\},$$

$$\cos\Delta = (\bar{r}_s^{\,0}, \bar{r}_y^{\,0}).$$

In scalar form the Eq. (2.19) is

$$\cos\Delta = \cos\varphi_s \cos\varphi_0 \cos(\lambda_s - \lambda_P) + \sin\varphi_s \sin\varphi_0 .$$

6. In Appendix 1, the satellite's system of equations of motion is given along with the control algorithm.

It is known that when an instantaneous impulse is applied, the corresponding thrust switches on at $v = \frac{3}{2}\pi$ (i.e., past $\frac{\pi}{2}$ until passing the encounter point). Therefore, as a first approximation for the switch-on time of the thrust we select this value t_{on_1} (it corresponds to $v_{on_1} = \frac{3}{2}\pi$). Upon rezeroing the functional (2.18) thrust cut off takes place when $v = v_{off_1}$, which corresponds to time t_{off_1}. Since the process of rotating the orbital plane in a central field possesses well-known symmetry, it is natural to select the time of thrust switch-on in the second approximation according to the following scheme.

Let the impulse $P_o = \int_0^{t_{off_1}} k_z d\tau$ be a known quantity; it is derived at the time of the rezeroing of the functional (2.18). We distribute this impulse equally from the point $v - v_{on_1}$ and select this as the criterion for the switch-on time of the propulsion based on the second approximation.

Then it is obvious that /12

$$\frac{P_o}{2} = \int_0^t K_z d\tau, \tag{2.20}$$

from which we derive, taking (2.4) into account,

$$\frac{P_o}{2} = \int_0^{\Delta t} \frac{K_o}{1 - B\tau} d\tau \tag{2.21}$$

or, solving equation (2.21) with respect to Δt, we obtain

$$\Delta t = \frac{1 - e^{-\frac{P_o B}{2K_o}}}{B}. \tag{2.22}$$

Taking into account the fact that

$$\Delta v_{on} = \nu_o \Delta t,$$

we obtain the algorithm for switching on in the second approximation

$$v_{on_2} = v_{on_1} - \Delta v_{on},$$

i.e.,

$$v_{on_2} = v_{on_1} - \frac{\dot{\nu}_o}{B}\left[1 - e^{-\frac{P_o B}{2K_o}}\right]. \tag{2.23}$$

We note that in the case of a constant thrust, when B = 0, we obtain

$$\lim_{B\to 0} \Delta t = \lim_{B\to 0} \frac{1 - e^{-\frac{P_0 B}{2K_0}}}{B} = \lim_{B\to 0} \frac{e^{\frac{P_0 B}{2K_0}} \cdot \frac{P_0}{2K_0}}{1} = \frac{P_0}{2K_0}, \tag{2.24}$$

which can be derived directly from (2.21).

It follows from (2.24) that

$$\Delta V = \frac{\nu_0}{2} \cdot \frac{P_0}{K_0},$$

and since the thrust is constant,

$$\Delta V = \Delta V_{on} = \frac{V_{off_1} - V_{on_1}}{2}, \tag{2.25}$$

where it is evident that $\quad V_{on_2} = V_{on_1} + \Delta V_{on},$

or /13

$$V_{on_2} = \frac{3\pi + V_{off_1} - V_{on_1}}{2}. \tag{2.26}$$

Having obtained the value of v_{on_2} from Eq. (2.23') or from Eq. (2.26), the calculation is repeated again until the rezeroing of the functional (2.18). The machine time expended in the calculation of the active section is not great (it does not exceed ~ 5 minutes).

The results of the calculation, even based on the simplified Eq. (2.26), give a fuel saving of ~30% and more for certain conditions of flight. We note that the time of active flight simultaneously decreases upon the calculation of the second approximation.

Algorithm for Rotating the Orbital Plane
in a Noncentral Field

7. The gravitational field of the earth is earth is taken with an accuracy of the first term of the expansion

$$U = \frac{\mu}{r}\left[1 + \varepsilon_1\left(1 - 3\sin^2\varphi\right)\right], \tag{3.1}$$

where

$$\varepsilon_1 = \varepsilon\left(\frac{R_0}{r}\right)^2,$$

$$R_0 = 6.37815 \cdot 10^6 \text{ M},$$

$$\varepsilon = 0.001623.$$

Prescribing the gravitational field of the earth with high accuracy does not have practical meaning as far as the set-up of the problem is concerned. We attempt to apply the scheme outlined above to derive an algorithm for rotating the orbital plane in a noncentral field for the problem of the motion of a satellite in a noncentral field.

As is well known: /14

$$\left.\begin{aligned} K_r &= -\varepsilon_1\left(1 - 3\sin^2 i\,\sin^2 u\right), \\ K_v &= -\varepsilon_1 \sin^2 i\,\sin 2u, \\ K_z &= -\varepsilon_1 \sin 2i\,\sin u. \end{aligned}\right\} \tag{3.2}$$

Taking into account (3.2) and replacing the variables in (2.9) instead of (2.1), we obtain the following system of equations

$$\left.\begin{aligned} &1.\ y' = z, \\ &2.\ z' = -y + \frac{1}{p}\left[1 - \varepsilon_1\left(1 - 3\gamma_1^2 + 2\frac{z}{y}\gamma_1\gamma_2\right)\right], \\ &3.\ p' = -\frac{4}{y}\varepsilon_1\gamma_1\gamma_2, \\ &4.\ \gamma_1' = \gamma_2, \\ &5.\ \gamma_2' = -\gamma_2 + \frac{\gamma_3}{py}\left(K_z - 2\varepsilon_1\gamma_1\gamma_3\right), \\ &6.\ \gamma_3' = -\frac{\gamma_2}{py}\left(K_z - 2\varepsilon_1\gamma_1\gamma_3\right), \end{aligned}\right\} \tag{3.3}$$

where here as earlier it is possible to write instead of the last equation on the basis of the existence of the integral (2.10). $\gamma_3^2 = 1 - \gamma_1^2 - \gamma_2^2$.

8. The system (3.3) together with the control system solves the proposed problem exactly. However, to derive the functional of cut-off in a noncentral field, it is scarcely convenient, since usually the equations of a satellite in a specific interval of time of the motion can be linearized relative to some reference orbit. Moreover, it is well known that if one gives a specific motion (i.e., precession) of the orbital plane with a given Ω_3, then the linearized equations have essentially the same form. Under the conditions of the proposed problem, people usually take

$$\Omega_3 = -\varepsilon_1 \cos i = \text{const}. \tag{3.4}$$ /15

with high accuracy. The physical meaning of such a linearization consists in the fact that the equations of motion of a satellite can be considered in the first approximation to be independent of its motion in the osculating plane and the motion of the osculating plane itself. However, such linearization is conveniently carried out in other variables but is not applicable for the variables γ_1, γ_2, and γ_3.

Based on the idea itself for setting up the problem, we would like to have at any time the analytic solution (3.3), only in this case we will be able to use effectively our prediction functional.

Two situations can arise: when the reference orbit is circular and when the reference orbit is elliptical; in the latter case the equations are somewhat more complicated.

For realistic limitations

$$\left| \frac{\delta y}{y_0} \right| \leq \varepsilon, \tag{3.5}$$

where

y_0 corresponds to the unperturbed motion of the satellite, and

δy is the deviation from this motion upon a perturbation of the trajectory in the case of a noncentral gravitational field.

One can separate an independent subsystem (see Appendix 2) from the system (3.3):

$$\left.\begin{aligned} \gamma_1' &= \gamma_2, \\ \gamma_2' &= -\gamma_1\left[1 + 2\frac{(R_0 y_0)^2}{P_0 y_0}\varepsilon\gamma_3^2\right], \\ \gamma_3^2 &= 1 - \gamma_1^2 - \gamma_2^2, \end{aligned}\right\} \tag{3.6}$$ /16

where

P_o is the parameter of the unperturbed orbit.

If the altitude of the circular orbit above the earth's surface is ~130 km, then the limitations (3.5) correspond to $|\delta y|_{max} = 10.5$ km.

We note that the algorithm for switching on the propulsion in a noncentral field remains the same as in a central field (see Section 6), since in practice

$$|\kappa_2| >> 2\varepsilon_1 \geqslant 2\varepsilon_1\gamma_1\gamma_3 \geqslant \varepsilon_1, \tag{3.7}$$

where according to Appendix 2

$$\varepsilon_1 = (R_0 y_0)^2\varepsilon = \mathrm{const}, \tag{3.8}$$

holds for circular orbits, $P_o y_o = 1$.

9. Based on the concept of setting up the problem, it is necessary for us to know the satellite's motion at any time after thrust cut-off, when the satellite is moving only under the effect of the gravitational field of the earth. This reduces to the solution of the following nonlinear system of equations:

$$\left.\begin{aligned} \gamma_1' &= \gamma_2, \\ \gamma_2' &= -\gamma_1(1 + 2\varepsilon_1\gamma_3^2), \\ \gamma_3^2 &= 1 - \gamma_1^2 - \gamma_2^2, \end{aligned}\right\} \tag{3.9}$$

where

ε_1 = const.

The reference orbit is taken to be circular.

Let the initial conditions for (3.9) have the following form:

$$\gamma_1(0) = \gamma_{10}, \quad \gamma_2(0) = \gamma_{20}. \tag{3.10}$$

These expressions are derived continuously as a result of the solution of the algorithm.

We perform the following obvious transformations:

$$\gamma_2' \gamma_2 = -\gamma_1 \gamma_1' (1 + 2\varepsilon_1 \gamma_3^2),$$
$$\gamma_3 \gamma_3' = -\gamma_1 \gamma_1' - \gamma_2 \gamma_2',$$

where $\gamma_3^1 = 2\varepsilon_4\gamma_4\gamma_2\gamma_3$ for $\gamma_3 \neq 0$; if $\gamma_3 = 0$, then we obtain a trivial case. It follows from the last expression that

$$\frac{d\gamma_3}{\gamma_3} = 2\varepsilon_1 \gamma_1 d\gamma_1, \tag{3.11}$$

and the variables are separated, giving

$$\ln \frac{\gamma_3}{C} = \varepsilon_1 \gamma_1^2,$$

from which

$$\gamma_3 = C e^{\varepsilon_1 \gamma_1^2}.$$

We determine the constant C:

$$\gamma_{30} = C e^{\varepsilon_1 \gamma_{10}^2}, \quad C = \gamma_{30} e^{-\varepsilon_1 \gamma_{10}^2};$$

we get

$$\gamma_3 = \gamma_{30} e^{\varepsilon_1(\gamma_1^2 - \gamma_{10}^2)}. \tag{3.12}$$

Substituting (3.12) into the three-equation system (3.9), we obtain

$$\gamma_1^2 + \gamma_2^2 + \gamma_{30}^2 e^{2\varepsilon_1(\gamma_1^2 - \gamma_{10}^2)} = 1, \tag{3.13}$$

or, taking into account the first equation of (3.9) /18

$$\gamma_1^2 + (\gamma_1')^2 + \gamma_{30}^2 e^{2\varepsilon_1(\gamma_1^2-\gamma_{10}^2)} = 1.$$

The variables are separated in the last equation, and the solution reduces to a quadrature:

$$\int \frac{d\gamma_1}{\sqrt{1-\gamma_1^2-\gamma_{20}^2 e^{2\varepsilon_1(\gamma_1^2-\gamma_{10}^2)}}} = \Delta v + C. \qquad (3.14)$$

We note that in the plane of the phase variables $\gamma_1\gamma_2$ the expression (3.13) represents an ellipse with first-order accuracy up to the terms ε_1. Actually, discarding terms in the expansion of a lower order of magnitude relative to ε_1, we obtain

$$\gamma_1^2 + \gamma_2^2 + \gamma_{30}^2\left[1+2\varepsilon_1(\gamma_1^2-\gamma_{10}^2)\right] \approx 1,$$

or

$$(1+2\varepsilon_1\gamma_{30}^2)\gamma_1^2 + \gamma_2^2 = 1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2),$$

from which the equation of the ellipse will have the following form:

$$\frac{\gamma_1^2}{a^2} + \frac{\gamma_2^2}{b^2} = 1, \qquad (3.15)$$

where

$$a^2 = \frac{1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2)}{1+2\varepsilon_1\gamma_{30}^2},$$

$$b^2 = 1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2).$$

When $\varepsilon_1 = 0$ and a = b, the expression (3.15) is converted into the equation of a circle with a radius

$$r = \sqrt{1-\gamma_{30}^2}.$$

In order not to calculate the constant C, we rewrite (3.14) in the form

$$\int_{\gamma_{10}}^{\gamma_1} \frac{d\gamma_1}{\sqrt{1-\gamma^2-\gamma_{30}^2 e^{2\varepsilon_1(\gamma^2-\gamma_{10}^2)}}} = \Delta V. \qquad (3.16)$$ ∠19

We expand the integrand of the expression into a Maclaurin series in the small parameter ε_1 to the order ε_1^2, inclusively

$$f(\varepsilon_1) = \frac{1}{\sqrt{1-\gamma^2-\gamma_{30}^2 e^{2\varepsilon_1(\gamma^2-\gamma_{10}^2)}}} \approx f(0)+f'(0)\varepsilon_1 + \frac{f''(0)}{2}\varepsilon_1^2, \qquad (3.17)$$

where

$$f(0) = \frac{1}{\sqrt{a^2-\gamma^2}},$$

$$f'(0) = \frac{\gamma_{30}^2(\gamma^2-\gamma_{10}^2)}{(a^2-\gamma^2)^{3/2}},$$

$$f''(0) = \frac{2\gamma_{30}^2(\gamma^2-\gamma_{10}^2)^2}{(a^2-\gamma^2)^{3/2}}\left[1+\frac{3}{2}\gamma_{30}^2\frac{1}{a^2-\gamma^2}\right],$$

$$a^2 = 1-\gamma_{10}^2.$$

We integrate the first two terms in Eq. (3.17):

$$\int_{\gamma_{10}}^{\gamma_1} \frac{d\gamma}{\sqrt{a^2-\gamma^2}} = \arcsin\frac{\gamma}{a}\Bigg|_{\gamma_{10}}^{\gamma_1} = \arcsin\frac{\gamma_1}{a} - \arcsin\frac{\gamma_{10}}{a}. \qquad (3.18)$$

Upon integrating the following expression

$$\gamma_{30}^2 \int_{\gamma_{10}}^{\gamma_1} \frac{\gamma^2-\gamma_{10}^2}{(a^2-\gamma^2)^{3/2}}\, d\gamma \qquad (3.19)$$

we make the following replacement of variables:

$$\gamma = a^2\sin\varphi,$$

$$d\gamma = a^2\cos\varphi\, d\varphi.$$

Then we write instead of (3.19)

$$\gamma_{30} a \int_{\varphi_0}^{\varphi} \frac{\sin^2\varphi - \frac{\gamma_{10}^2}{a^2}}{\cos\varphi}\, d\varphi = \gamma_{30} a \left[\int_{\varphi_0}^{\varphi} \frac{\sin^2\varphi}{\cos\varphi}\, d\varphi - \frac{\gamma_{10}^2}{a^2}\int_{\varphi_0}^{\varphi}\frac{d\varphi}{\cos\varphi}\right], \qquad (3.20)$$ /20

where $\varphi_0 = \arcsin \frac{\gamma_{10}}{a^2}$, $\varphi = \arcsin \frac{\gamma}{a^2}$;

$$\int_{\varphi_0}^{\varphi} \frac{\sin^2\varphi}{\cos\varphi}\, d\varphi = -\sin\varphi + \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\Bigg|_{\varphi_0}^{\varphi} =$$

$$= -\frac{\gamma_1 - \gamma_{10}}{a^2} + \ln \frac{\operatorname{tg}\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\operatorname{tg}\left(\frac{\pi}{4} + \frac{\varphi_0}{2}\right)} = \qquad (3.21)$$

$$= -\frac{\gamma_1 - \gamma_{10}}{a^2} + \frac{1}{2} \ln \frac{\left(1 - \frac{\gamma_{10}}{a^2}\right)\left(1 + \frac{\gamma_1}{a^2}\right)}{\left(1 + \frac{\gamma_{10}}{a^2}\right)\left(1 - \frac{\gamma_1}{a^2}\right)},$$

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\cos\varphi} = \ln \operatorname{tg}\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\Bigg|_{\varphi_0}^{\varphi} = \ln \frac{\operatorname{tg}\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)}{\operatorname{tg}\left(\frac{\varphi_0}{2} + \frac{\pi}{4}\right)} = \frac{1}{2} \ln \frac{\left(1 - \frac{\gamma_{10}}{a^2}\right)\left(1 + \frac{\gamma_1}{a^2}\right)}{\left(1 + \frac{\gamma_{10}}{a^2}\right)\left(1 - \frac{\gamma_1}{a^2}\right)}. \qquad (3.22)$$

Taking into account (3.16), (3.18), (3.20), (3.21), and (3.22), we obtain

$$\Delta V = \arcsin \frac{\gamma_1}{a} - \arcsin \frac{\gamma_{10}}{a} +$$
$$+ \left\{-\frac{\gamma_{30}}{a}(\gamma_1 - \gamma_{10}) + \frac{1}{2} \ln \frac{\left(1 - \frac{\gamma_{10}}{a^2}\right)\left(1 + \frac{\gamma_1}{a^2}\right)}{\left(1 + \frac{\gamma_{10}}{a^2}\right)\left(1 - \frac{\gamma_1}{a^2}\right)} \left[\gamma_{30} a - \frac{\gamma_{10}^2}{a^2}\right]\right\} + o(\varepsilon_1). \qquad (3.23)$$

Here

$$|o(\varepsilon_1)| = \int_{\gamma_{10}}^{\gamma_1} \left|\frac{f''(0)}{2}\varepsilon_1^2\right| d\gamma_1 = \varepsilon_1^2 \gamma_{30}^2 \int_{\gamma_{10}}^{\gamma_1} \left|\frac{(\gamma^2 - \gamma_{10}^2)^2}{(a^2-\gamma^2)^{3/2}}\left(1 + \frac{3}{2}\gamma_{30}^2 \frac{1}{a^2-\gamma^2}\right)\right| d\gamma \leq$$

$$\leq \gamma_{30}^2 \varepsilon_1^2 \int_{\gamma_{10}}^{\gamma_1} \left|\frac{(\gamma^2 - \gamma_{10}^2)^2}{(a^2-\gamma^2)^{3/2}}\right| \cdot \left|1 + \frac{3}{2}\gamma_{30}^2 \frac{1}{a^2-\gamma^2}\right| d\gamma .$$

One can show that

$$\left|\frac{(\gamma_1^2 - \gamma_{10}^2)^2}{(a^2-\gamma^2)^{3/2}}\right| \leq 3{,}1 \quad ; \quad \left|1 + \frac{3}{2}\gamma_{30}^2 \frac{1}{a^2-\gamma^2}\right| \leq 1 + \frac{3}{2}\left(\frac{\gamma_{30}}{a}\right)^2,$$

from which $$o(\varepsilon_1) \leq 3{,}1\,\gamma_{30}^2\left[1+\frac{3}{2}\left(\frac{\gamma_{30}}{a}\right)^2\right]\varepsilon_1^2(\gamma_1-\gamma_{10}),$$ /21

i.e., Eq. (3.23) is valid in the restricted interval $|\gamma_1 - \gamma_{10}| \leq \frac{\pi}{2}$ with an accuracy of the order of $o(\varepsilon_1)$.

10. We attempt to find a simpler form of solution than the one obtained in the preceding section.

We expand the integrand in Eq. (3.16) in the following manner:

$$\left[1-\gamma^2-\gamma_{30}^2 e^{2\varepsilon_1(\gamma^2-\gamma_{10}^2)}\right]^{-1/2} = \left\{1-\gamma^2-\gamma_{30}^2\left[1+2\varepsilon_1(\gamma^2-\gamma_{10}^2)+o(\xi_1)+\ldots\right]\right\}^{-1/2} =$$
$$= \left[1-\gamma_{30}^2(1+2\varepsilon_1\gamma_{10}^2)-\gamma^2(1+2\varepsilon_1\gamma_{30}^2)\right]^{-1/2}\cdot(1+\xi)^{-1/2}, \qquad (3.24)$$

where
$$o(\varepsilon_1) = 2(\gamma^2-\gamma_{10}^2)\cdot\varepsilon_1^2+\ldots\ldots,$$
$$\xi = \frac{-\gamma_{30}^2\cdot o(\varepsilon_1)}{1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2)-\gamma^2(1+2\varepsilon_1\gamma_{30}^2)}.$$

One can show that $|\xi| < 1$. Actually,
$$\gamma_{10}^2 = 1-\gamma_{20}^2-\gamma_{30}^2 \leq 1-\gamma_{30}^2.$$

We expand the denominator of ξ:
$$1-\gamma_{30}^2+2\varepsilon_1\gamma_{30}^2\gamma_{10}^2-\gamma^2-2\varepsilon_1\gamma^2\gamma_{30}^2 \geq \gamma_{10}^2-\gamma^2+2\varepsilon_1\gamma_{30}^2(\gamma_{10}^2-\gamma_{30}^2),$$

from which
$$|\xi| < \left|\frac{o(\xi_1)}{\gamma_{10}^2-\gamma^2}\right| + \left|\frac{o(\xi_1)}{2\varepsilon_1\gamma_{30}^2(\gamma_{10}^2-\gamma_{30}^2)}\right|,$$

taking into account the symbols above, we obtain /22
$$|\xi| < |2(\gamma_{10}^2-\gamma^2)\varepsilon_1^2| < 1.$$

We expand the last bracket in (3.24) into a series and obtain
$$\left[1-\gamma^2-\gamma_{30}^2 e^{2\varepsilon_1(\gamma^2-\gamma_{10}^2)}\right]^{-1/2} = \qquad (3.25)$$
$$= \left[1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2)-\gamma^2(1+2\varepsilon_1\gamma_{30}^2)\right]^{-1/2}\cdot\left[1-\frac{1}{2}\xi+o(\xi)\right],$$

where

$$o(\xi) = \frac{1\cdot 3}{2\cdot 4}\xi^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\xi^3 + \ldots$$

If one writes down the following approximate equality

$$\left[1-\gamma^2-\gamma_{30}^2 e^{2\varepsilon_1(\gamma^2-\gamma_{10}^2)}\right]^{-1/2} \approx \left[1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2)-\gamma^2(1+2\varepsilon_1\gamma_{30}^2)\right]^{-1/2},$$

then according to (3.25) it is correct with an accuracy m, where

$$|m| < \frac{1}{2}\left|\xi_{max}\right| < \frac{1}{2}2\varepsilon_1^2\left|\gamma_{10}^2-\gamma^2\right|_{max} < \varepsilon_1^2 . \tag{3.26}$$

According to Eq. (3.16), we obtain an estimate for the integral

$$\left|\int_{\gamma_{10}}^{\gamma_1}\frac{d\gamma}{\sqrt{1-\gamma^2-\gamma_{30}^2 e^{2\varepsilon_1(\gamma^2-\gamma_{10}^2)}}}\right| \approx (1\pm\varepsilon_1^2)\left|\int_{\gamma_{10}}^{\gamma_1}\frac{d\gamma}{\sqrt{1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2)-(1+2\varepsilon_1\gamma_{30}^2)\gamma^2}}\right| .$$

We introduce the following symbols:

$$K_0^2 = 1+2\varepsilon_1\gamma_{30}^2 ,$$

$$A^2 = 1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2) .$$

Integrating the right part of the last equation, we obtain

$$\Delta V = \frac{1}{K_0}\arcsin\frac{K_0\gamma}{A}\Bigg|_{\gamma_{10}}^{\gamma_1} = \frac{1}{K_0}\left[\arcsin\frac{K_0\gamma_1}{A} - \arcsin\frac{K_0\gamma_{10}}{A}\right], \tag{3.27}$$

or /23

$$\gamma_1 = \frac{A}{K_0}\sin\left[K_0\Delta V + \arcsin\frac{K_0\gamma_{10}}{A}\right] . \tag{3.28}$$

Furthermore, it is evident that

$$\gamma_1' = A\cos\left[K_0\Delta V + \arcsin\frac{K_0\gamma_{10}}{A}\right] . \tag{3.29}$$

It follows from (3.28) that when $\Delta v = 0$, $\gamma_1 = \gamma_{10}$, but from (3.29)

$$\gamma_2 = A\cos\left[\arcsin\frac{K_0\gamma_{10}}{A}\right] = A\cos\left[\arccos\sqrt{1-\frac{K_0^2\gamma_{10}^2}{A^2}}\right] = \sqrt{A^2-K_0^2\gamma_{10}^2}. \tag{3.30}$$

If the expressions for A and K_o are expanded, then we obtain the identity:

$$\gamma_{20}^2 = A^2 - K_0^2\gamma_{10}^2 = 1-\gamma_{30}^2(1-2\varepsilon_1\gamma_{10}^2)-\gamma_{10}^2(1+2\varepsilon_1\gamma_{30}^2) = 1-\gamma_{30}^2-\gamma_{10}^2.$$

11. We solve the system (3.9) by another method. For this purpose, we apply the functional method of Newton. We have

$$\gamma_1'' = f(\gamma_1, \gamma_1'), \quad \gamma_1(0) = \gamma_{10}, \quad \gamma_1'(0) = \gamma_{10}', \tag{3.31}$$

where

$$f(\gamma_1, \gamma_1') = -\left[1+2\varepsilon_1\left(1-\gamma_1^2-(\gamma_1')^2\right)\right]\gamma_1.$$

We introduce the functional which is the distance to a desired point in the functional space $\gamma_1(v)$:

$$I = \int_0^{\Delta v}\left[\gamma_1''-f(\gamma_1,\gamma_1')\right]^2 d\tau, \tag{3.32}$$

where

$\Delta v = v - v_o$,

v_o is the coordinate corresponding to the current time.

The approach of the functional I to zero upon fulfillment of the initial conditions denotes the tendency of $\gamma_1(v)$ to the desired solution $\gamma_1{}^*(v)$. /24

As in any method of successive approximation, it is important to choose successfully the approximation $\gamma_1^{(0)}(v)$. We choose this solution in a form similar to (3.28):

$$\begin{aligned}\gamma_1^{(0)} &= M_0\sin(K_0\Delta v + L_0),\\ \gamma_1'^{(0)} &= M_0K_0\cos(K_0\Delta v + L_0),\end{aligned} \tag{3.33}$$

where

$$M_0^2 = \gamma_{10}^2 + \frac{\gamma_{20}^2}{K_0^2},$$

$$tg\, L_0 = \frac{K_0 \gamma_{10}}{\gamma_{20}};$$

$$K_0^2 = 1 + 2\varepsilon_1 \left[1 - \gamma_{10}^2 - (\gamma_{10}')^2\right].$$

We note that in comparison with the preceding section we have here

$$M_0 = \frac{A}{K_0},$$

$$\sin L_0 = \frac{K_0 \gamma_{10}}{A},$$

$$\cos L_0 = \sqrt{1 - \frac{K_0^2 \gamma_{10}^2}{A}} = \frac{\sqrt{A^2 - K_0^2 \gamma_{10}^2}}{A} = \frac{\gamma_{20}}{A},$$

$$tg\, L_0 = \frac{K_0 \gamma_{10}}{\gamma_{20}},$$

$$M_0^2 = \frac{A^2}{K_0^2} = \frac{\gamma_{20}^2 + K_0^2 \gamma_{10}^2}{K_0^2} = \gamma_{10}^2 + \frac{\gamma_{20}^2}{K_0^2}.$$

We represent the succeeding approximation in the form

$$\gamma_1^{(1)}(v) = \gamma_1^{(0)}(v) + \delta\gamma_1(v),$$

where

$$\|\delta\gamma_1\| << \|\gamma_1^{(0)}\|, \quad \delta\gamma_1(0) = 0, \quad \delta\gamma_1'(0) = 0.$$

The variation in the functional I caused by the variation $\delta\gamma_1'(v)$, is

$$\delta J = 2\int_0^{\Delta v} \left[\gamma_1^{''(0)} - f^{(0)}\right] \cdot \left[\delta\gamma_1'' - \frac{\partial f^{(0)}}{\partial \gamma_1'}\delta\gamma_1' - \frac{\partial f^{(0)}}{\partial \gamma_1}\delta\gamma_1\right] d\tau. \qquad (3.34)$$

Here $\gamma_1^{''(0)}, f^{(0)}, \frac{\partial f^{(0)}}{\partial \gamma_1'}, \frac{\partial f^{(0)}}{\partial \gamma_1}$ are functions of Δv, calculated with /25

$\gamma_1 = \gamma_4^{(0)}(v)$.

Furthermore, based on a well-known procedure, we select from among all the variations the following:

$$\delta\gamma_1'' - \frac{\partial f^{(0)}}{\partial \gamma_1'}\delta\gamma_1' - \frac{\partial f^{(0)}}{\partial \gamma_1}\delta\gamma_1 = -\lambda\left(\gamma_1''^{(0)} - f^{(0)}\right),$$
$$\delta\gamma_1(0) = 0, \qquad \delta\gamma_1'(0) = 0. \tag{3.35}$$

We introduce the function $\eta(v)$:

$$\eta(v) = \frac{\delta\gamma_1(v)}{\lambda}.$$

Then

$$\gamma_1^{(1)}(v) = \gamma_1^{(0)}(v) + \lambda\eta(v),$$

and we obtain instead of (3.35)

$$\eta'' - \frac{\partial f^{(0)}}{\partial \gamma_1'}\eta' - \frac{\partial f^{(0)}}{\partial \gamma_1}\eta = -\gamma_1''^{(0)} + f^{(0)},$$
$$\eta(0) = 0, \qquad \eta'(0) = 0. \tag{3.36}$$

and so, according to (3.31) we obtain

$$\frac{\partial f^{(0)}}{\partial \gamma_1'} = 4\varepsilon_1\gamma_1^{(0)}\gamma_1'^{(0)},$$
$$\frac{\partial f^{(0)}}{\partial \gamma_1} = 6\varepsilon_1\left(\gamma_1^{(0)}\right)^2 + 2\varepsilon_1\left(\gamma_1'^{(0)}\right)^2 - 1 - 2\varepsilon_1. \tag{3.37}$$

Substituting (3.33) into (3.37), we obtain

$$\frac{\partial f^{(0)}}{\partial \gamma_1'} = 2\varepsilon_1 M_0^2 \kappa_0 \sin\left[2(\kappa_0\Delta v + L_0)\right],$$
$$\frac{\partial f^{(0)}}{\partial \gamma_1} \approx -1 - 2\varepsilon_1 + 2\varepsilon_1 M_0^2\left[2 - \cos 2(\kappa_0\Delta v + L_0)\right] \doteq \tag{3.38}$$
$$= -1 - 2\varepsilon_1(1 - 2M_0^2) - 2\varepsilon_1 M_0^2 \cos\left[2(\kappa_0\Delta v + L_0)\right];$$

the last equation is correct to an accuracy of $\sim\varepsilon_1^2$.

We determine the right-hand side of Eq. (3.36)

$$-\gamma_1''^{(0)} + f^{(0)} = M_0 \kappa_0^2 \sin(\kappa_0 \Delta v + L_0) - \tag{3.39}$$
$$- [1 + 2\varepsilon_1 (1 - M_0^2)] M_0 \sin(\kappa_0 \Delta v + L_0) =$$
$$= -M_0 \sin(\kappa_0 \Delta v + L_0) 2\varepsilon_1 (\gamma_{10}')^2 \frac{2\varepsilon_1 \gamma_{30}^2}{K_0^2} =$$
$$= -4\varepsilon_1^2 (\gamma_{10}')^2 \frac{\gamma_{30}^2}{K_0^2} M_0 \sin(\kappa_0 \Delta v + L_0).$$

And so

$$\| -\gamma_1''^{(0)} + f^{(0)} \| < \varepsilon_1^2 . \tag{3.40}$$

The error for the same variables is

$$\| \eta \| \leqslant \varepsilon_1^2 \frac{(v - v_0)^2}{2} .$$

The maximum value of $v - v_o = \frac{\pi}{2}$, from which

$$\| \eta_{max} \| \leqslant \varepsilon_1^2 \frac{\pi^2}{8} \approx 1.24 \varepsilon_1^2 .$$

If we choose $\lambda = I$, then already as a result of the first iteration

$$\| \delta\gamma_1 \|_{max} = \| \eta \|_{max} = 1.24 \varepsilon_1^2 .$$

The order of magnitude of the accuracy of the solution ($\sim \varepsilon_1^2$) with respect to the first approximation agrees with that obtained in Section 10 with the Eq. (3.26) and amounts to $\sim 1 \times 10^{-6}$.

Such agreement is explained by the fact that γ_3 is a slowly varying quantity ($\gamma_3 = \cos i$). Therefore, the first approximation to the solution, taken as $\gamma_3 = \gamma_{30}$, already gives a sufficiently accurate solution. It is well known that for all iterative methods it is important to specify sufficiently accurately the first point (in functional space). In this connection it is clear why the natural expansion in the small parameter ε_1, which was presented in Section 9, is less successful. /27

12. The discussion above concerning an estimate of the first approximation shows that its error does not exceed a quantity $\sim O(\varepsilon_1) \approx \varepsilon_1^2$. This error should not be admitted into the calculation, since the problem itself is set up with an

accuracy not exceeding the quantity ε_1. And so, a solution of the system (3.9) can be written in the following form:

$$\left.\begin{aligned} \gamma_1 &= M_0 \sin(\kappa_0 \Delta v + L_0), \\ \gamma_2 &= M_0 \kappa_0 \cos(\kappa_0 \Delta v + L_0), \\ \gamma_3 &= 1 - M_0^2 \left[1 + 2\varepsilon_1 \gamma_{30}^2 \cos^2(\kappa_0 \Delta v + L_0)\right], \end{aligned}\right\} \tag{3.41}$$

where

$$\kappa_0^2 = 1 + 2\varepsilon_1 \gamma_{30}^2, \qquad \operatorname{tg} L_0 = \frac{\kappa_0 \gamma_{10}}{\gamma_{20}},$$
$$M_0^2 = \gamma_{10}^2 + \frac{\gamma_{20}^2}{\kappa_0^2}.$$

Upon introducing the equations for the cut-off functional in a noncentral field, complications appear which are connected with the fact that in finding the prediction time it is now impossible to use a single-valued relation between Δu_y and Δv_y, since now $w = \text{const}$.

If we use the derived solution (3.41), then we can establish this connection, using the relation (2.14), which will now have the following form:

$$\operatorname{tg}(u + \Delta u_y) = \frac{\gamma_1(v, \Delta v_y)}{\gamma_2(v, \Delta v_y)}, \tag{3.42}$$

where γ_{10}, γ_{20}, γ_{30} are the current values of γ_1, γ_2, γ_3, which are calculated /28
at the same time on the trajectory

$$\operatorname{tg}(u + \Delta u_y) = \frac{1}{\kappa_0} \operatorname{tg}(\kappa_0 \Delta v_y + L_0). \tag{3.43}$$

Since

$$\operatorname{tg}(u + \Delta u_y) = \frac{\sin(u + \Delta u_y)}{\sqrt{1 - \sin^2(u + \Delta u_y)}} = \frac{\gamma_{1y}}{\sqrt{1 - \gamma_{1y}^2 - \gamma_3^2}},$$

then we obtain instead of (3.43)

$$\operatorname{tg}(\kappa_0 \Delta v_y + L_0) = \frac{\kappa_0 \gamma_{1y}}{\sqrt{1 - \gamma_{1y}^2 - \gamma_3^2}},$$

from which

$$\Delta V_\gamma = \frac{\operatorname{arctg}\left(K_0 \frac{\gamma_1}{\gamma}\right) + \operatorname{arctg} \frac{K_0 \gamma_{1\gamma}}{\sqrt{1-\gamma_{1\gamma}^2-\gamma_\gamma^2}}}{K_0} \tag{3.44}$$

The complete system of equations for the algorithm for rotating the orbital plane in a noncentral field is presented in Appendix 3.

We note that when $\varepsilon_1 = 0$, this system of equations can serve as an algorithm for rotation of the orbital plane in a central field.

In Appendix 3.1 the expression for the propulsion cut-off functional is written in a different form than in Appendix 1.1, which appears to be more acceptable here.

13. The system of equations in Appendix 3.1 is the simplest and most convenient in practical applications; in addition, it solves with sufficient accuracy the problem posed. An estimate of this accuracy was made above. /29

The largest error which one could call systematic arises for the relation

$$t = f(v). \tag{3.45}$$

We now estimate this error. The inaccuracy in the knowledge of (3.45) is expressed by the errors upon rezeroing the functional of the expression 10 in Appendix 3.1, where all the quantities are considered as arguments of v, and the values of λ_p are calculated on the basis of the expression 8 of the same system, whose errors are characteristic of (3.45) and vice versa.

Earlier, we wrote that

$$t' = \frac{1}{\sqrt{\mu p}\; y_\gamma^2}, \tag{3.46}$$

We take, as usual,

$$t_0' = \frac{1}{\sqrt{\mu p_0}\; y_{\gamma 0}^2}, \tag{3.47}$$

where

P_o and y_o were determined above.

We find the logarithmic derivatives in (3.46)

$$\ln t' = -\left[\frac{1}{2}\ln\mu + \frac{1}{2}\ln p + 2\ln y\right], \tag{3.48}$$

furthermore, the derivatives of the Eq. (3.48) at the point t_o is

$$\frac{dt'}{t'_o} = -\left[\frac{1}{2}\,\frac{dp}{p_o} + 2\,\frac{dy}{y_o}\right]. \tag{3.49}$$

Now, one can write the linearized equation for (3.46) /30

$$\frac{dt}{dv} = t' \approx t'_o + (\delta t)' = t'_o\left[1 + \frac{(\delta t)'}{t'_o}\right]. \tag{3.50}$$

Evidently,

$$\left|\frac{(\delta t)'}{t'_o}\right| = \left|\frac{1}{2}\,\frac{\delta p}{p_o} + 2\,\frac{\delta y}{y_o}\right| \leq \frac{1}{2}\left|\frac{\delta p}{p_o}\right| + 2\left|\frac{\delta y}{y_o}\right|. \tag{3.51}$$

Taking into account Appendix 2.2 and Appendix 2.8, we obtain the following estimate:

$$\left|\frac{(\delta t)'}{t'_o}\right| \leq \frac{1}{2}\pi\varepsilon_1 + 2\varepsilon_1 \approx 3.5\,\varepsilon. \tag{3.52}$$

From the equation

$$\lambda_p = \lambda_o + \frac{\Omega_3}{\dot{\nu}_o}\,v = \lambda_o + \Omega_3 t'_o v \tag{3.53}$$

it follows that the error in linearization is

$$\delta\lambda_p \approx \Omega_3 t'_o \cdot 3.5\,\varepsilon v. \tag{3.54}$$

Assuming that the time of flight corresponds to $v = \pi/2$, we obtain

$$|\delta\lambda_p| \leq 0.2 \cdot 0.0016 \cdot \frac{\pi}{2} \approx 0.314\,\varepsilon. \tag{3.55}$$

This quantity corresponds to an error at the surface of the earth of ~ 3 km.

As was indicated above, all the remaining errors are less than the one derived here. Therefore, there is no sense in searching for a more accurate solution of the equations of motion with such information. The error of 3 km can be reduced if one knows more accurately the parameters of the orbit at the instant control begins, which in its turn implies a requirement for a more accurate description of the equations.

Appendix 1.

Combining the series of equations presented in the first section, one can describe the algorithm for rotating the orbital plane in a central field in the following form: /31

$$
\left.
\begin{array}{ll}
1. & \gamma_1' = \gamma_2, \\
2. & \gamma_2' = -\gamma_1 + K_z\gamma_3, \\
3. & \gamma_3^2 = 1 - \gamma_1^2 - \gamma_2^2, \\
4. & K_z = \dfrac{K_0}{1 - B\tau}, \\
5. & \tau = \left\{ \begin{array}{ll} 0, & t < \tau_{on(i)} \\ t - t_{on(i)}, & \tau_{on(i)} \le t \le \tau_{off} \\ t_{off} - t_{on(i)}, & t > \tau_{off} \end{array} \right\} (i = 1;2) \\
6. & t' = \dfrac{1}{\dot{\vartheta}_0}, \\
7. & \Omega' = K_z \dfrac{\gamma_1}{1 - \gamma_3^2}, \\
8. & \lambda_P = \lambda_0 + \dfrac{\Omega_3}{\dot{\vartheta}_0} v, \\
9. & u = v + w, \\
10. & w = \operatorname{arctg} \dfrac{\gamma_1}{\gamma_2} - v, \\
11. & \lambda_S = \Omega + \operatorname{arctg}\left(\gamma_3 \dfrac{\gamma_1}{\gamma_2}\right), \\
12. & \lambda_B = \Omega + \operatorname{arctg}\left(\gamma_{1z} \dfrac{\gamma_3}{\gamma_2}\right), \\
13. & \Delta v_y = 2\pi - u + \arcsin\left(\dfrac{\gamma_{1z}}{\sqrt{1 - \gamma_3^2}}\right), \\
14. & \lambda_B - \lambda_y - \dfrac{\Omega_3}{\dot{\vartheta}_0} \Delta v_y = 0, \\
15. & \cos\Delta = \sqrt{(1 - \gamma_1^2)(1 - \gamma_{1z}^2)} \cdot \cos(\lambda_c - \lambda_y) + \gamma_1 \gamma_{1z}.
\end{array}
\right\} \quad \text{(App. 1)}
$$

Here $\gamma_{1z} = \sin\varphi_0 = \text{const}.$

Appendix 2.

We consider the conditions under which it is possible to separate the system /32
(3.3) into two independent parts with an accuracy of the order of $\sim \varepsilon_1^{2}$.

$$
\left.
\begin{aligned}
&1.\quad y' = z,\\
&2.\quad z' = -y + \frac{1}{p}\left[1 - \varepsilon_1\left(1 - 3\gamma_1^2 + 2\frac{z}{y}\gamma_1\gamma_2\right)\right],\\
&3.\quad p' = -\frac{4}{y}\varepsilon_1\gamma_1\gamma_2,\\
&4.\quad \gamma_1' = \gamma_2,\\
&5.\quad \gamma_2' = -\gamma_1\left[1 + \frac{2\varepsilon_1\gamma_3^2}{py}\right],\\
&6.\quad \gamma_3' = -\frac{2\varepsilon_1}{py}\gamma_1\gamma_2\gamma_3.
\end{aligned}
\right\}
\qquad \text{(App. 2.1)}
$$

The system (App. 2.1) differs from (3.3) in the fact that in it $k_z = 0$, in accordance with the idea of the problem being considered. We assume that the deviation of the satellite from the unperturbed orbit satisfies the condition

$$\left|\frac{\delta y}{y_0}\right| \leq \varepsilon, \qquad \text{(App. 2.2)}$$

where

y_0 corresponds to the unperturbed motion.

Taking into account (App. 2.2), we obtain

$$\varepsilon_1 = \varepsilon\left(R_0 y\right)^2 \approx \varepsilon\left(R_0 y_0\right)^2\left[1 + 2\frac{\delta y}{y_0}\right],$$

from which we get with an accuracy of the order of $\sim \varepsilon^2$.

$$\varepsilon_1 = \left(R_0 y_0\right)^2 \varepsilon. \qquad \text{(App. 2.3)}$$

It is obvious that if the quantity Py varies little in the specified time interval and one can neglect its variations, then Eqs. 4, 5, and 6 can be solved independently of the remaining equations (App. 2.1). /33

$$\frac{1}{py} \approx \frac{1}{p_0 y_0}\left[1 - \frac{\delta(py)}{p_0 y_0}\right] = \frac{1}{p_0 y_0}\left[1 - \frac{\delta p}{p_0} - \frac{\delta y}{y_0}\right]. \qquad \text{(App. 2.4)}$$

To estimate these terms, we integrate Eq. (3) in (App. 2.1)

$$\frac{p'}{p_0} = -\frac{4}{p_0 y}\varepsilon_1 \gamma_1 \gamma_2 \approx -\frac{4}{p_0 y_0}\left(1 - \frac{\delta y}{y_0}\right)\varepsilon (R_0 y_0)^2 \left(1 + 2\frac{\delta y}{y_0}\right)\gamma_{10}\gamma_{20}.$$

We denote by the index "1" the first approximation

$$\begin{aligned} \frac{\delta p_1}{p_0} &\approx -4\varepsilon \int \frac{(R_0 y_0)^2}{p_0 y_0}\left(1 - \frac{\delta y}{y_0}\right)\left(1 + 2\frac{\delta y}{y_0}\right)\gamma_{10}\gamma_{20}\, dv, \\ \left|\frac{\delta p_1}{p_0}\right| &\leqslant 4\varepsilon \int \left|\frac{(R_0 y_0)^2}{p_0 y_0}(1-\varepsilon)(1+2\varepsilon)\gamma_{10}\gamma_{20}\right| dv \leqslant , \\ &\leqslant 4\varepsilon \int \left|\frac{(R_0 y_0)^2}{p_0 y_0}\right| \cdot \left|1 + \varepsilon - 2\varepsilon^2\right| \cdot \left|\gamma_{10}\gamma_{20}\right| dv. \end{aligned} \qquad \text{(App. 2.5)}$$

We make the following remarks with regard to the last inequality.

1. In every case, for an actual trajectory of the motion the parameter of the orbit P_0 is always larger than the average radius of the earth R_0, i.e., $P_0 > R_0$.

Then

$$\left|\frac{(R_0 y_0)^2}{p_0 y_0}\right| = \left|\frac{R_0^2 y_0}{p_0}\right| < \left|R_0 y_0\right|.$$

In its turn,

$$y_0 = 1/r_0$$

or

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$$\left| R_0 y_0 \right| \doteq \left| \frac{R_0}{r_0} \right| < 1 ,$$

where r_o is the radius vector of the unperturbed orbit, which is always larger than the average radius of the earth according to the idea of the problem.

Finally we obtain,

$$\left| \frac{(R_0 y_0)^2}{P_0 \gamma_0} \right| < 1 . \qquad \text{(App. 2.6)}$$

2. We note that, based on the conditions of the problem,

$$\gamma_{10}^2 + \gamma_{20}^2 + \gamma_{30}^2 = 1 ,$$

this follows, for example, from Eq. (2.10).

It is completely evident that

$$\gamma_{10}^2 + \gamma_{20}^2 \leq 1 ,$$

from which

$$\gamma_{10} \gamma_{20} \leq \gamma_{10} \sqrt{1 - \gamma_{10}^2} .$$

We find the extreme of this equation

$$\frac{\partial}{\partial \gamma_{10}} \left[\gamma_{10} \sqrt{1 - \gamma_{10}^2} \right] = 0 , \qquad \gamma_{10}^2 = \frac{1}{2} \quad \text{и} \quad \gamma_{10} \gamma_{20} \leq \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} .$$

Thus,

$$\left| \gamma_{10} \gamma_{20} \right| \leq \frac{1}{2} . \qquad \text{(App. 2.7)}$$

3. The expression $\left| 1 + \varepsilon - 2\varepsilon^2 \right|$ already takes into account terms of $O(\varepsilon)$. The integration range should at the maximum not exceed $\pi/2$ according to the idea of the problem, i.e.

$$\left| v - v_0 \right| \leq \frac{\pi}{2} .$$

Taking into account the remarks made above, we obtain instead of (App. 2.5)

$$\left| \frac{\delta P_1}{P_0} \right| \leq 2\varepsilon (1 + \varepsilon - 2\varepsilon^2) \cdot \int dv = 2\varepsilon (1 + \varepsilon - 2\varepsilon^2)(v - v_0) \leq \pi \varepsilon (1 + \varepsilon - 2\varepsilon^2) .$$

Since there is no sense taking into account the terms $\sim \varepsilon^2$ in the problem, then /35

$$\left|\frac{\delta p_1}{p_0}\right| \leq 2\varepsilon(v-v_0) \leq \pi\varepsilon, \qquad \text{(App. 2.8)}$$

and the error allowed in this case does not exceed

$$\sim \pi\varepsilon^2 = 8.26\cdot 10^{-6} = 0.826 10^{-5}.$$

We turn to equation (App. 214)

$$\frac{1}{py} \leq \frac{1}{p_0 y_0}\left[1-2\varepsilon(v-v_0)-\varepsilon\right] = \frac{1}{p_0 y_0}\left[1-\varepsilon\left(2(v-v_0)+1\right)\right]. \qquad \text{(App. 2.9)}$$

Under the conditions of our problem

$$\inf_{v-v_0\to\frac{\pi}{2}} \frac{1}{py} = \frac{1}{p_0 y_0}\left[1-\varepsilon(\pi+1)\right] \approx \frac{1-3{,}14\varepsilon}{p_0 y_0}. \qquad \text{(App. 2.10)}$$

In Eq. 5 of (App. 2.1) we consider the following term:

$$2\varepsilon_1 \frac{\gamma_3^2}{py} \approx 2\varepsilon(R_0 y_0)^2(1+2\varepsilon)\frac{\gamma_3^2}{p_0 y_0}(1-3{,}14\varepsilon) \approx 2\varepsilon\gamma_3^2\frac{(R_0 y_0)^2}{p_0 y_0},$$

from which we obtain instead of Eq. 5

$$\gamma_2' = -\gamma_1\left[1+2\varepsilon_1\gamma_3^2\right],$$

where

$$\varepsilon_1 = (R_0 y_0)^2\varepsilon;$$

for circular orbits

$$p_0 y_0 = 1 \quad \text{и} \quad \varepsilon_1 = (R_0 y_0)^2\varepsilon = const < \varepsilon.$$

Appendix 3.

The equations of motion in the case of the rotation of the orbital plane /36
in a noncentral field have the following form.

$$
\left.
\begin{aligned}
&1.\quad \gamma_1' = \gamma_2,\\
&2.\quad \gamma_2' = -\gamma_1 + \gamma_3\left(K_z - 2\varepsilon_1\gamma_1\gamma_3\right),\\
&3.\quad \gamma_3^2 = 1 - \gamma_1^2 - \gamma_2^2,\\
&4.\quad K_z = \frac{K_0}{1 - B\tau},\\
&5.\quad \tau = \left\{\begin{array}{lll} 0, & t < \tau_{on(i)} \\ t - t_{on(i)}, & \tau_{on(i)} \le t \le \tau_{off} \\ t_{off} - t_{on(i)}, & t > \tau_{off} \end{array}\right\}(i = 1; 2)\\
&6.\quad t' = \frac{1}{\dot{\nu}_0},\\
&7.\quad \Omega' = \frac{\gamma_1}{1-\gamma_3^2}\left(K_z - 2\varepsilon_1\gamma_1\gamma_3\right),\\
&8.\quad \lambda_p = \lambda_0 + \frac{\Omega_3}{\dot{\nu}_0}v,\\
&9.\quad \lambda_s = \Omega + \operatorname{arctg}\left(\frac{\gamma_1\gamma_3}{\gamma_2}\right),\\
&10.\quad \Delta V_y = \frac{\operatorname{arctg}\left[\frac{K_0 K_{1z}}{\sqrt{1-\gamma_{1z}^2-\gamma_3^2}}\right] + \operatorname{arctg}\left[K_0\frac{\gamma_1}{\gamma_2}\right]}{K_0},\\
&11.\quad \cos\Delta = \sqrt{\left(1-\gamma_1^2\right)\left(1-\gamma_{1z}^2\right)}\cdot\cos\left(\lambda_c - \lambda_y\right) + \gamma_1\gamma_{1z},
\end{aligned}
\right\}\qquad \text{(App. 3.1)}
$$

where $K_0^2 = 1 + 2\varepsilon_1\gamma_3^2$,

$\dot{\nu}_0 = \frac{1}{t_0'} = \sqrt{\mu P_0}\, y_0^2$,

$\gamma_{1z} = \sin\varphi_0 = const$,

$\varepsilon_1 = (R_0 y_0)^2\varepsilon = const$,

$\varepsilon = 0,001623$,

$y_0 = \frac{1}{r_0}$

r_0 – is the radius of the circular reference orbit.

In the system (App. 3.1) Eq.(11) can be taken as the functional according to /37
which the propulsion cut-off takes place. The value of $v = v_{\text{cut-off}}$ is obtained upon rezeroing of the equation

$$\cos\Delta - 1 = 0 \qquad \text{(App. 3.2)}$$

and the value of the argument is taken equal to $v + \Delta v_y$, i.e., at the assumed time of passage over the point P. In such a form, the system (App. 3.1) is not suitable for calculation, since the values of the quantities must be taken at the predicted time $v + \Delta v_y$. For this we use the analytic expressions for the variables γ_1, γ_2, and γ_3 obtained above in (3.30), which we substitute into Eqs. (7), (9), and (11) of the system (App. 3.1).

Eq. (7) has the following form ($K_z = 0$):

$$\Omega' = -2\varepsilon_1 \frac{\gamma_1^2\gamma_3}{1-\gamma_3^2} = -2\varepsilon_1 \frac{\sin^2(K_0\Delta v + L_0)\sqrt{1 - M_0^2\left[1 + 2\varepsilon_1\gamma_{30}^2\cos^2(K_0\Delta v + L_0)\right]}}{1 + 2\varepsilon_1\gamma_{30}^2\cos^2(K_0\Delta v - L_0)}. \qquad \text{(App. 3.3)}$$

Here, it is necessary to recall that γ_{30} is the initial value corresponding to the current value of γ_3, calculated directly from Eq. (3) of the system (App. 3.1); therefore, the index "0" will be dropped from here on.

Instead of Eq. (9) we have the following:

$$\lambda_c = \Omega + \mathrm{arctg}\left[\frac{\mathrm{tg}(K_0\Delta v + L_0)}{K_0}\sqrt{1 - M_0^2\left[1 + 2\varepsilon_1\gamma_3^2\cos^2(K_0\Delta v + L_0)\right]}\right]. \qquad \text{(App. 3.4)}$$

Eq. (11) for the cut-off functional now has the following form (see App. 3.2): /38

$$\sqrt{(1-\gamma_{1y}^2)\left[1 - M_0^2\sin^2(K_0\Delta v + L_0)\right]}\cdot\cos(\lambda_c - \lambda_y) + \gamma_{1y}M_0\sin(K_0\Delta v + L_0) - 1 = 0. \qquad \text{(App. 3.5)}$$

In all three of the last relations (3, 4, and 5) one should substitute the quantity $\Delta v = \Delta v_y$ instead of Δv, which is found from Eq. (10) of the system (App. 3.1).

To find the functional, it is necessary to know Ω, for which it is necessary to integrate (App. 3.3). To simplify the calculations we expand the right-hand side of this equation in powers of ε_1:

$$\Omega^1 = -2\varepsilon_1 \sin^2(K_0\Delta V + L_0)\left[\sqrt{1-M_0^2} - \gamma_3^2\cos^2(K_0\Delta V + L_0)\left(\frac{M_0^2}{\sqrt{1-M_0^2}} + 2\right)\varepsilon_1 + \ldots\right].$$

If as before one restricts the terms in ε_1 to the first order of magnitude, then we obtain

$$\Omega^1 \approx -2\sqrt{1-M_0^2}\cdot\varepsilon_1 \sin^2(K_0\Delta V + L_0). \qquad \text{(App. 3.6)}$$

Having integrated this equation, we obtain

$$\Omega = -\varepsilon_1\frac{\sqrt{1-M_0^2}}{K_0}\left[K_0\Delta V + L_0 - \frac{1}{2}\sin 2(K_0\Delta V + L_0)\right] \qquad \text{(App. 3.7)}$$

or

$$\Omega = -\varepsilon_1\sqrt{1-M_0^2}\cdot\frac{K_0\Delta V + L_0}{K_0}\left[1 - \frac{\sin 2(K_0\Delta V + L_0)}{2(K_0\Delta V + L_0)}\right]. \qquad \text{(App. 3.8)}$$

Having taken account of these considerations, we obtain the following algorithm for the rotation of the orbital plane in a noncentral field:

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$$\left.\begin{array}{ll}
1. & \gamma_1' = \gamma_2, \\
2. & \gamma_2' = -\gamma_1 + \gamma_3(K_2 - 2\varepsilon_1\gamma_1\gamma_3), \\
3. & \gamma_3^2 = 1 - \gamma_1^2 - \gamma_2^2, \\
4. & K_2 = \dfrac{K_0}{1-\beta\tau}, \\
5. & \tau = \begin{cases} 0, & t < \tau_{on(i)} \\ t - t_{on(i)}, & \tau_{on(i)} \leq t \leq \tau_{off} \\ t_{off} - t_{on(i)}, & t > \tau_{off} \end{cases} \quad (i=1;2)
\end{array}\right\} \qquad \text{(App. 3.9)}$$

6. $t = t_0' v$,

7. $\lambda_p = \lambda_0 + \Omega_3 t_0' v$,

8. $K_0 \Delta v_y - L_0 = \operatorname{arctg} \dfrac{K_0 K_{13}}{\sqrt{1 - \gamma_{13}^2 - \gamma_3^2}}$,

9. $\Omega' = \dfrac{\gamma_1}{1 - \gamma_3^2} \left(K_z - 2\varepsilon_1 \gamma_1 \gamma_3 \right)$,

10. $\tilde{\Omega} = \Omega - \varepsilon_1 \sqrt{1 - M_0^2} \cdot \dfrac{K_0 \Delta v_y + L_0}{K_0} \left[1 - \dfrac{\sin 2 (K_0 \Delta v_y + L_0)}{2 (K_0 \Delta v_y + L_0)} \right]$,

11. $\tilde{\lambda}_S = \tilde{\Omega} + \operatorname{arctg} \left[\dfrac{\operatorname{tg}(K_0 \Delta v_y - L_0)}{K_0} \sqrt{1 - M_0^2 \left[1 + 2\varepsilon_1 \gamma_3^2 \cos^2 (K_0 \Delta v_y + L_0) \right]} \right]$,

12. $\sqrt{(1 - \gamma_{13}^2) \left[1 - M_0^2 \sin^2 (K_0 \Delta v_y + L_0) \right]} \cdot \cos (\tilde{\lambda}_0 - \lambda_y) + \gamma_{13} M_0 \sin (K_0 \Delta v_y + L_0) - 1 = 0$,

where $K_0^2 = 1 + 2\varepsilon_1 \gamma_3^2$,

$t_0' = \dfrac{1}{\sqrt{\mu p_0}\; y_0^2}$,

$\gamma_{13} = \sin \varphi_0 = \text{const}$,

$\varepsilon_1 = (R_0 y_0)^2 \varepsilon = \text{const}$,

$\varepsilon = 0{,}001623$,

$y_0 = \dfrac{1}{r_0}$,

$M_0^2 = \gamma_1^2 + \dfrac{\gamma_3^2}{K_0^2}$.

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